An Exact Analytical Expression of the Shear Coefficient in the Mindlin Plate Equation

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PREFACE

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13. ABSTRACT (Maximum 200 words)

This report derives an exact analytical expression of the shear coefficient in the Mindlin plate equation. The Mindlin plate equation is set equal to the thick plate equation, and the result is a closed-form expression of the shear coefficient at all wavenumbers and frequencies. A numerical example is included to show the variation of the shear coefficient. It is shown that the shear coefficient is extremely dependent on wavenumber and only slightly dependent on frequency. Shear coefficients derived in other work are compared favorably to the values calculated by this new method at the plate flexural wave response.

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IN THE MINDLIN PLATE EQUATION

INTRODUCTION

The Mindlin plate equation is a model of a thin plate with the addition of rotary inertia and shear effects on the dynamics of the plate response.[1] Incorporated in the Mindlin plate equation is a shear coefficient, which is an adjustment parameter in thick plate, beam, and shell equations of motion that is included to compensate for stress distribution in the cross-sectional shape of the object. It is sometimes designated the Timoshenko[2,3] shear coefficient, a concept introduced by Timoshenko to account for the non-uniform shear stress distribution in a beam. Timoshenko estimated the shear coefficient as a function of Poisson's ratio in the beam.[3] Cowper[4] also estimated the shear coefficient as a function of Poisson's ratio, although he derived a slightly different expression. Over the years, it has become evident in beam theory that the shear coefficient is theoretically dependent on more than Poisson's ratio.[5-8] The problem of determining the shear coefficient in a Mindlin plate equation has been addressed by Stephen,[9] who determined what he called a "best shear coefficient" based on matching a mode of the Mindlin plate theory to the exact Rayleigh-Lamb frequency equation for the flexural wave response.

The above theories are all based on excitation of the flexural wave in a structure, and the corresponding shear coefficients are determined at the specific wavenumber and frequency of the flexural wave. They do not account for the shear coefficient as a function of all wavenumbers and frequencies, which is a response and excitation condition that exists in structures that are loaded by turbulent boundary layers or acoustical forces.

This report derives an exact analytical expression of the shear coefficient in the Mindlin plate equation subjected to planewave excitation at any wavenumber and frequency. This is accomplished by computing the displacement field of the plate using the Mindlin plate equation and the thick plate equation, and then setting them equal to each other. Because the shear coefficient is explicit in the Mindlin plate equation and implicit in the thick plate equation, it can be solved for as a function of wavenumber, frequency, and plate parameters. A numerical example is included to depict the dependence of the shear coefficient on wavenumber and frequency. It is shown that the shear coefficient is only slightly dependent on frequency and extremely dependent on wavenumber of excitation. Comparisons of previous analytical expressions are also included in the numerical example to illustrate how other theories compare to the one derived here.

SYSTEM MODELS AND THEORETICAL SHEAR COEFFICIENT

Two system models are developed: one contains the shear coefficient explicitly and the other contains the shear coefficient implicitly. The first system model is that of a Mindlin plate whose governing equation is[10]

$$\left(\nabla^2 - \frac{\rho}{k\mu}\frac{\partial^2}{\partial t^2}\right)\left(D\nabla^2 - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\right)u(x,t) + \rho h\frac{\partial^2 u(x,t)}{\partial t^2} = \left(1 - \frac{D\nabla^2}{k\mu h} + \frac{\rho h^2}{12k\mu}\frac{\partial^2}{\partial t^2}\right)f(x,t), \quad (1)$$

where k is the shear coefficient, $u(x,t)$ is the displacement of the plate in the z-direction, $f(x,t)$ is the force distribution on the plate, ρ is the density, μ is the shear modulus, h is the thickness, x is spatial location, t is time, ∇ is the spatial gradient operator, and D is equal to

$$D = \frac{Eh^3}{12(1-\upsilon^2)}, \quad (2)$$

where E is Young's modulus, and υ is Poisson's ratio. The system is modeled as infinitely long with a continuous forcing function varying in time and space; thus, the displacement and forcing function terms are written as

$$u(x,t) = U(p,\omega)\exp(i\omega t)\exp(ipx), \quad (3)$$

and

$$f(x,t) = F(p,\omega)\exp(i\omega t)\exp(ipx), \quad (4)$$

where ω is angular frequency and p is wavenumber with respect to the x-axis. Solving the transfer function response of displacement divided by input force yields

$$\frac{U(p,\omega)}{F(p,\omega)} = \frac{-12\mu hk - 12Dp^2 + \rho h^3\omega^2}{(12D\mu hp^4 - \mu\rho h^4\omega^2 p^2 - 12\mu\rho h^2\omega^2)k - 12D\rho h\omega^2 p^2 + \rho^2 h^4\omega^4}, \quad (5)$$

or

$$\frac{U(p,\omega)}{F(p,\omega)} = \frac{ak+b}{ck+d}, \quad (6)$$

where

$$a = -12\mu h, \quad (7)$$

$$b = -12Dp^2 + \rho h^3 \omega^2, \qquad (8)$$

$$c = 12 D \mu h p^4 - \mu \rho h^4 \omega^2 p^2 - 12 \mu \rho h^2 \omega^2, \qquad (9)$$

and

$$d = -12D\rho h \omega^2 p^2 + \rho^2 h^4 \omega^4. \qquad (10)$$

The second model is derived from the equations of motion[11] of a solid medium, governed by

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\nabla\nabla \bullet \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \qquad (11)$$

where λ and μ are the Lamé constants, $\bullet$ denotes a vector dot product, and $\mathbf{u}$ is the Cartesian coordinate displacement vector of the plate. Assuming harmonic response in space and time, equation (11) can be manipulated; the resulting expression[12] is the displacement in the z-direction at the middle location of the plate divided by the incident force and is written as

$$\frac{U_z(p, -h/2, \omega)}{F(p,\omega)} = \frac{\Phi(p, -h/2, \omega)}{\mu\Delta(p,\omega)}, \qquad (12)$$

where

$$\begin{aligned}
\Phi(k_x, -h/2, \omega) = {} & 8\alpha^2 \beta p^2 (\beta^2 - p^2)\sin(\alpha h/2)\cos^2(\beta h/2) \\
& + 4\alpha p^2 (\beta^2 - p^2)^2 \sin(\beta h/2)\cos^2(\alpha h/2) \\
& + 16\alpha^2 \beta p^4 \sin(\alpha h/2)\cos(\alpha h/2)\cos(\beta h/2) \\
& + 2\alpha(\beta^2 - p^2)^3 \sin(\beta h/2)\cos(\alpha h/2)\cos(\beta h/2),
\end{aligned} \qquad (13)$$

and

$$\begin{aligned}
\Delta(p,\omega) = {} & -8\alpha\beta p^2 (\beta^2 - p^2)^2 [\cos(\alpha h)\cos(\beta h) - 1] \\
& + [(\beta^2 - p^2)^4 + 16\alpha^2 \beta^2 p^4]\sin(\alpha h)\sin(\beta h).
\end{aligned} \qquad (14)$$

In equations (13) and (14), α is the modified wavenumber associated with the dilatational wave and is expressed as

$$\alpha = \sqrt{k_d^2 - p^2}, \qquad (15)$$

where k_d is the dilatational wavenumber equal to ω / c_d, where c_d is the dilatational wavespeed (m/sec); β is the modified wavenumber (rad/m) associated with the shear wave and is expressed as

$$\beta = \sqrt{k_s^2 - p^2} \ , \tag{16}$$

where k_s is the shear wavenumber (rad/m) equal to ω / c_s, where c_s is the shear wavespeed (m/sec). The relationship between the wavespeeds (c_d and c_s) and the plate's Lamé constants (λ and μ) is determined by

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}} \ , \tag{17}$$

and

$$c_s = \sqrt{\frac{\mu}{\rho}} \ . \tag{18}$$

The shear coefficient is now determined by equating equations (6) and (12) and solving for k, which results in

$$k(p, \omega) = \frac{d\Phi - b\mu\Delta}{a\mu\Delta - c\Phi} \ . \tag{19}$$

When the system response is at the flexural resonance, the determinant of equation (12) is zero and the corresponding shear coefficient calculated from equation (19) is

$$k_f(p, \omega) = \frac{-d}{c} = \frac{12D\rho\omega^2 p^2 - \rho^2 h^3 \omega^4}{12D\mu p^4 - \mu\rho h^3 \omega^2 p^2 - 12\mu\rho h\omega^2} \ . \tag{20}$$

It is noted that this shear coefficient is not only a function of wavenumber and frequency, but also of Young's modulus, shear modulus, Poisson's ratio, and the thickness and density of the plate.

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NUMERICAL EXAMPLE

A numerical example is now analyzed to investigate the behavior of the shear coefficient calculated using equations (19) and (20). The parameters of the plate are listed in table 1. Figure 1 is a plot of the shear coefficient versus wavenumber and frequency as determined using equation (19). The parabolic line on the plot is the flexural wavenumber location calculated by finding the maximum value of the displacement in wavenumber at each analysis frequency. The weak line originating at the origin and ending at $f = 10,000$ Hz and $p = 75.4$ rad/m is the plate wave dynamics propagating through the analysis. This plate wave wavenumber can be predicted by

$$p_p = \omega \sqrt{\frac{\rho(1-\upsilon^2)}{E}} \;.$$ (21)

The thick plate equation of motion contains the plate wave dynamics while the Mindlin plate equation does not; thus, accounting for a modeling difference around the plate wavenumber. The result of this model mismatch is that the theoretical shear coefficient factor will not be accurate in the region around the plate wavenumber. Figure 2 is a cut of figure 1 in wavenumber at 4000 Hz, which illustrates the dependence of the shear coefficient on wavenumber. The square marker ($\square$) is the plate wavenumber and the round marker ($\bigcirc$) is the flexural wavenumber. Figure 3 is a cut of figure 1 in frequency at the flexural wavenumber. The shear coefficient estimate derived by Timoshenko[3] is indicated by the symbol $\times$ and is given by the wavenumber and frequency-independent equation

$$k = \frac{5+5\upsilon}{6+5\upsilon} .$$ (22)

The star symbol ($*$) is the shear coefficient derived by Stephen[9] and is equal to

$$k = \frac{5}{6-\upsilon} ,$$ (23)

and the plus symbol ($+$) is the shear coefficient estimate derived by Cowper[4] and is written as

$$k = \frac{10+10\upsilon}{12+11\upsilon} .$$ (24)

These shear coefficients are shown at low frequency because they were derived using long wavelength assumptions. It is clear that these estimates are close to the exact solution derived in equation (20) for wave propagation at (or near) the flexural wave propagation wavenumber and frequency. They do not account for wave propagation at other wavenumbers.

Table 1. Plate Parameters Used for Numerical Example

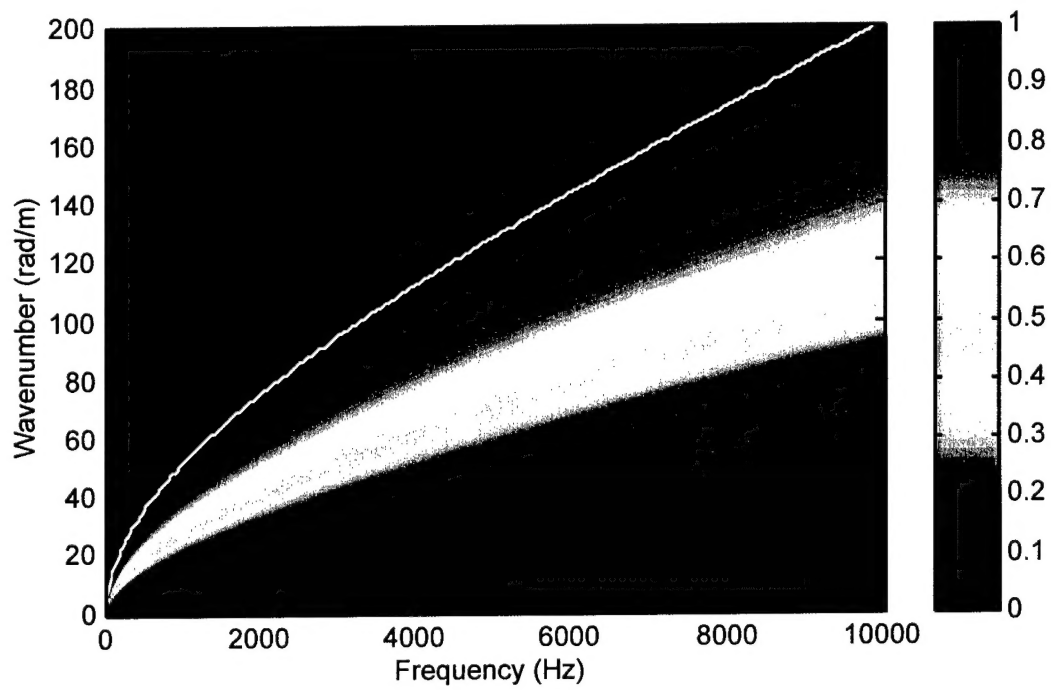| Parameter	Value
Thickness, h	0.01 m
Young's Modulus, E	7.0e8 N/m^2
Shear Modulus, G	2.5e8 N/m^2
Poisson's Ratio, υ	0.4 (dimensionless)
Density, ρ	1200 kg/m^3

Figure 1. Shear Coefficient Versus Wavenumber and Frequency for the Numerical Example
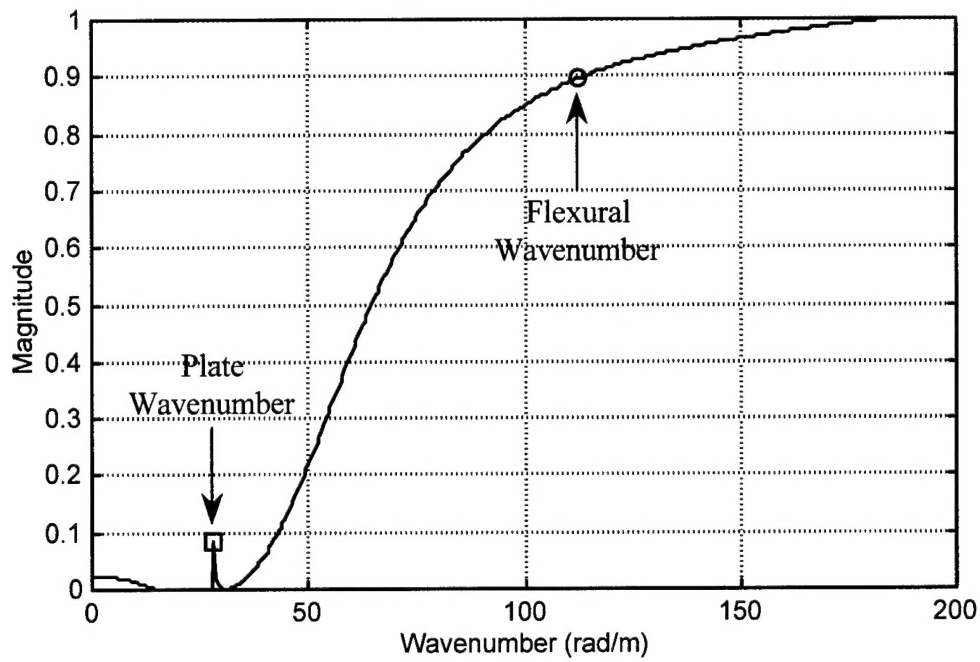
Figure 2. Shear Coefficient Versus Wavenumber at 4000 Hz for the Numerical Example
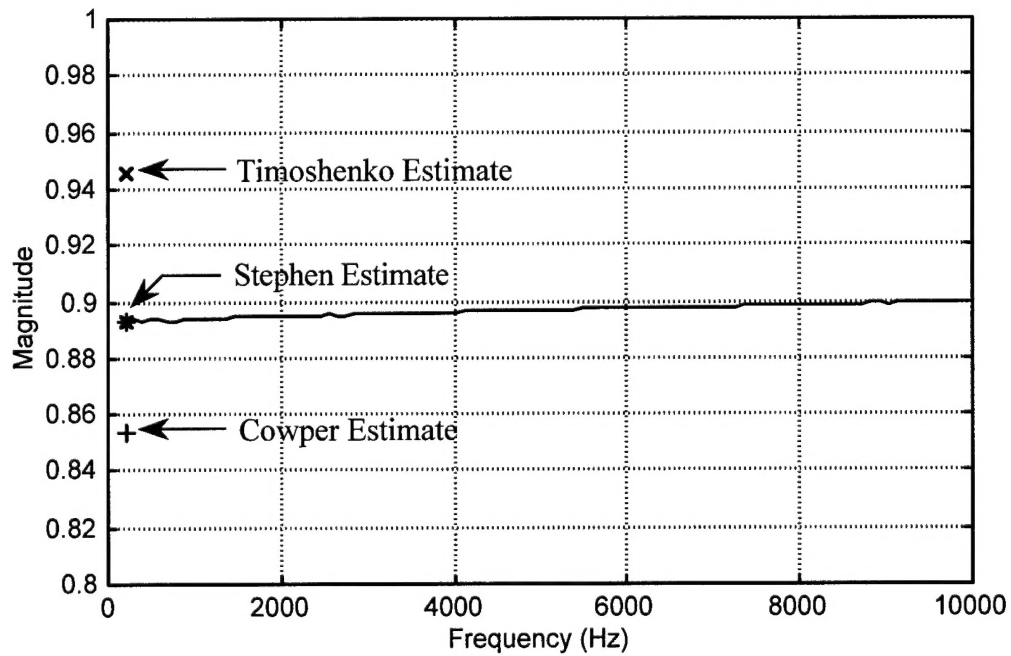

*Figure 3. Shear Coefficient Versus Frequency at Flexural Wave Resonance
for the Numerical Example*

The most interesting aspect of this analysis is that the shear coefficient is highly dependent on the wavenumber. This dependency was previously unknown; however, the fact that the dynamics of the system are dependent on the wavenumber of the excitation is not a new concept. In this type of analysis, low wavenumber excitation generally results in a structural response that is composed of primarily dilatational waves that contain the majority of the energy. (At zero wavenumber, the response is entirely a dilatational wave response.) Shear effects are secondary until the wavenumber of the excitation becomes moderate. Previous work has typically consisted of exciting the flexural wave in a beam, measuring response, and then back-calculating the shear correction factor. It has not consisted of exciting the structure at all wavenumbers and then determining the correction factor. (It should be noted that broadband wavenumber excitation of a structure is not easy to implement.)

CONCLUSIONS

The theoretical shear coefficient for a Mindlin plate has been derived as an analytical expression. It is shown that this term is dependent on wavenumber, frequency, Young's modulus, shear modulus, Poisson's ratio, density, and thickness of the plate. Numerical simulations showed that the shear coefficient is extremely dependent on wavenumber but only slightly dependent on frequency. Previous shear coefficient expressions are close to the analytical expression derived here around the flexural wave wavenumber of the plate.

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